



TITLE:

# HilbG(C4) and crepant resolutions of certain abelian groups in $SL(4, \mathbb{C})$

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# Hilb<sup>G</sup>(C<sup>4</sup>) and crepant resolutions of certain abelian groups in SL(4,C)

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## Background

## Question

Let  $G$  be a finite subgroup of  $SL(n, \mathbb{C})$ , then the quotient  $\mathbb{C}^n/G$  has a Gorenstein canonical singularity. When does  $\mathbb{C}^n/G$  have a **crepant resolution**?

- When  $n = 2$ , the quotient  $\mathbb{C}^2/G$  has a hypersurface singularity which is called a **rational double point** or **ADE singularity**.  $\mathbb{C}^2/G$  has the minimal resolution (It is a crepant resolution).
  - In the case  $n = 3$ , it is known that  $\mathbb{C}^3/G$  has crepant resolutions.
  - However, in higher dimension,  $\mathbb{C}^n/G$  does not always have crepant resolutions, and few examples of crepant resolutions are known.
- In this poster, we will show several examples of crepant resolutions in  $SL(4, \mathbb{C})$  by  $\text{Hilb}^G(\mathbb{C}^4)$

## Definition

A resolution  $f: Y \rightarrow X$  is called a crepant resolution if the adjunction formula  $K_Y = f^*K_X + \sum_{i=1}^n a_i D_i$  is satisfy  $a_i = 0$  for all  $i$

## Definition

$\text{Hilb}^G(\mathbb{C}^n) = \{I \subset \mathbb{C}[x_1, \dots, x_n] \mid I : G\text{-invariant ideal, } \mathbb{C}[x_1, \dots, x_n]/I \cong \mathbb{C}[G]\}$

- When  $n = 2$ ,  $\text{Hilb}^G(\mathbb{C}^2)$  is a crepant resolution for any finite subgroup in  $SL(2, \mathbb{C})$
- In the case  $n = 3$ , for any finite subgroup  $SL(3, \mathbb{C})$ ,  $\text{Hilb}^G(\mathbb{C}^3)$  is one of crepant resolutions.
- If  $n \geq 4$ , the relationship between  $\text{Hilb}^G(\mathbb{C}^n)$  and crepant resolutions is not well known.

## Crepant resolution as toric varieties

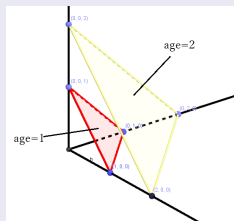
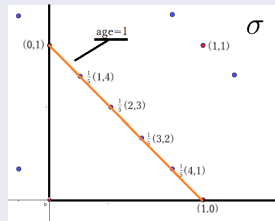
$G$  denote a finite abelian subgroups of  $SL(n, \mathbb{C})$ . Any  $g \in G$  is of the form  $g = \begin{pmatrix} \varepsilon_r^{a_1} & & 0 \\ & \ddots & \\ 0 & & \varepsilon_r^{a_n} \end{pmatrix}$ , where  $\varepsilon_r$  is a primitive  $r$ th root of unity.

Then we can write  $g = \frac{1}{r}(a_1, \dots, a_n)$ . Also, we define  $\bar{g} = \frac{1}{r}(a_1, \dots, a_n) \in \mathbb{R}$ . Let  $N := \mathbb{Z}^n + \mathbb{Z}\bar{g}$  be a free  $\mathbb{Z}$ -module of rank  $n$ ,  $M$  be the dual  $\mathbb{Z}$ -module of  $N$ , and  $\sigma$  be the region of  $\mathbb{R}^n$  whose all entries are non-negative.

Then the toric variety determined by  $\sigma$  is isomorphic to  $\mathbb{C}^n/G$

## Remark

When a cone  $\sigma$  to corresponding to  $\mathbb{C}^n/G$  can be subdivided into  $\Delta$  corresponding to smooth variety by lattice points of  $\text{age}(g) = 1$ , then the toric variety determined by  $\Delta$  is a crepant resolution of  $\mathbb{C}^n/G$ , where we define  $\text{age}(g) = \frac{1}{r} \sum_{i=1}^n a_i$ .



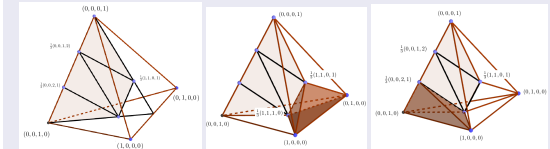
When  $n = 2$ , lattice points of  $\text{age} = 1$  are on straight line. If  $n = 3$ , they are on triangle.

So we consider **tetrahedron** in the case of  $SL(4, \mathbb{C})$

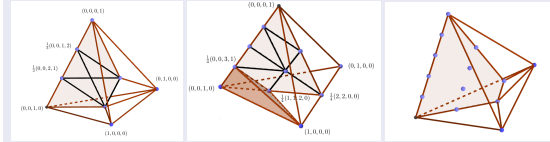
## Main Result

### Result 1

Let  $r \geq 2$ ,  $G = \langle \frac{1}{r}(1, 1, 0, r-2), \frac{1}{r}(0, 0, 1, r-1) \rangle$ . Then  $\mathbb{C}^4/G$  has crepant resolutions. If  $r$  is even, then  $\text{Hilb}^G(\mathbb{C}^4)$  is one of crepant resolutions. When  $r$  is odd,  $\text{Hilb}^G(\mathbb{C}^4)$  is blow-up of certain crepant resolutions.



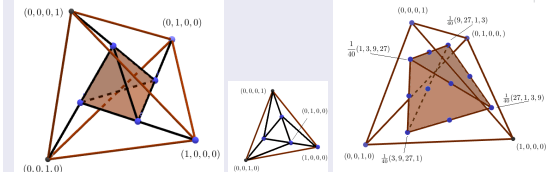
One of crepant resolutions for  $G = \langle \frac{1}{3}(1, 1, 0, 1), \frac{1}{3}(0, 0, 1, 2) \rangle$  and some examples of cone.



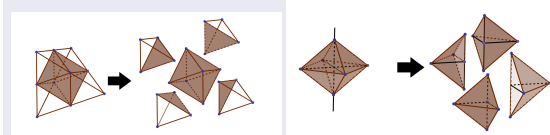
$\text{Hilb}^G(\mathbb{C}^4)$  for  $G = \langle \frac{1}{5}(1, 1, 0, 1), \frac{1}{5}(0, 0, 1, 2) \rangle$  and lattice points of  $\text{age}(g)=1$  of  $G = \langle \frac{1}{4}(1, 1, 0, 2), \frac{1}{4}(0, 0, 1, 3) \rangle$  and  $G = \langle \frac{1}{5}(1, 1, 0, 3), \frac{1}{5}(0, 0, 1, 4) \rangle$

### Result 2

Let  $r = 1 + k + k^2 + k^3$ ,  $G = \langle \frac{1}{r}(1, k, k^2, k^3) \rangle$ . Then  $\mathbb{C}^4/G$  has crepant resolutions. If  $k = 2$ , then  $\text{Hilb}^G(\mathbb{C}^4)$  is one of crepant resolutions for  $\mathbb{C}^4/G$ . When  $k \geq 3$ ,  $\text{Hilb}^G(\mathbb{C}^4)$  is blow-up of certain crepant resolutions.



Lattice point of  $\text{age} = 1$  for  $G = \langle \frac{1}{15}(1, 2, 4, 8) \rangle$  and  $G = \langle \frac{1}{40}(1, 3, 9, 27) \rangle$   $G = \langle \frac{1}{15}(1, 2, 4, 8) \rangle$  is a 4 dimensional version of  $G = \langle \frac{1}{3}(1, 2, 4) \rangle \subset SL(3, \mathbb{C})$

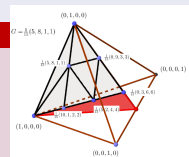


Subdivision of inside tetrahedron of  $G = \langle \frac{1}{40}(1, 3, 9, 27) \rangle$  seems like 'orange-slice'

$\text{Hilb}^G(\mathbb{C}^4)$  is a half of each pyramid of orange-slice and other cones are the same as crepant resolution.

### Other examples

If  $G = \langle \frac{1}{15}(5, 8, 1, 1) \rangle$  or  $G = \langle \frac{1}{15}(1, 8, 3, 3) \rangle$ , then  $\mathbb{C}^4/G$  has crepant resolutions.  $\text{Hilb}^G(\mathbb{C}^4)$  is one of crepant resolutions for  $\mathbb{C}^4/G$ . The lattice points of  $G$  are on a blue triangle and are similar to lattice points of  $\frac{1}{6}(1, 2, 3) \subset SL(3, \mathbb{C})$



### Reference

[HIS] T.Hayashi, Y.Ito, Y.Sekiya, Existence of crepant resolutions, Advanced Study in Pure Mathematics, vol.74 (2017), 185-202.